Evaluation of Bfloat16, Posit, and Takum Arithmetics in Sparse Linear Solvers

Laslo Hunhold[1] James Quinlan[2]

[1]Parallel and Distributed Systems Group, University of Cologne, Germany

[2]Department of Computer Science, University of Southern Maine, Portland, ME, USA

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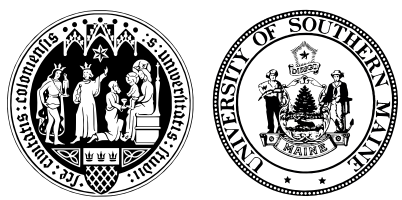

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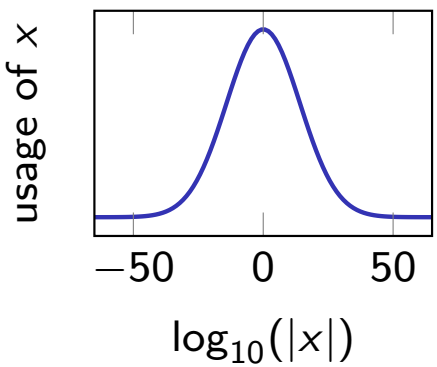

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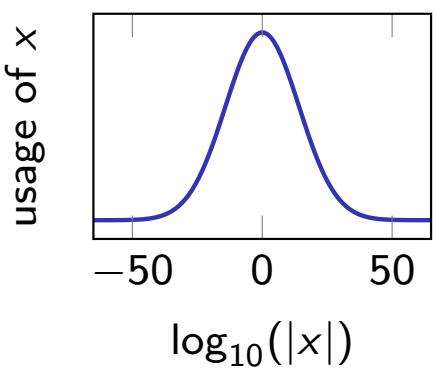

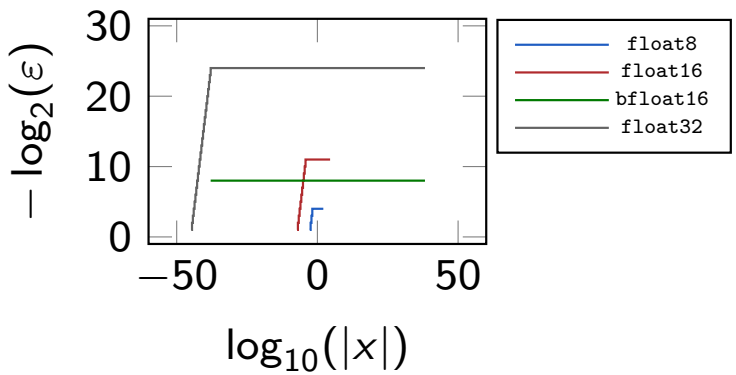

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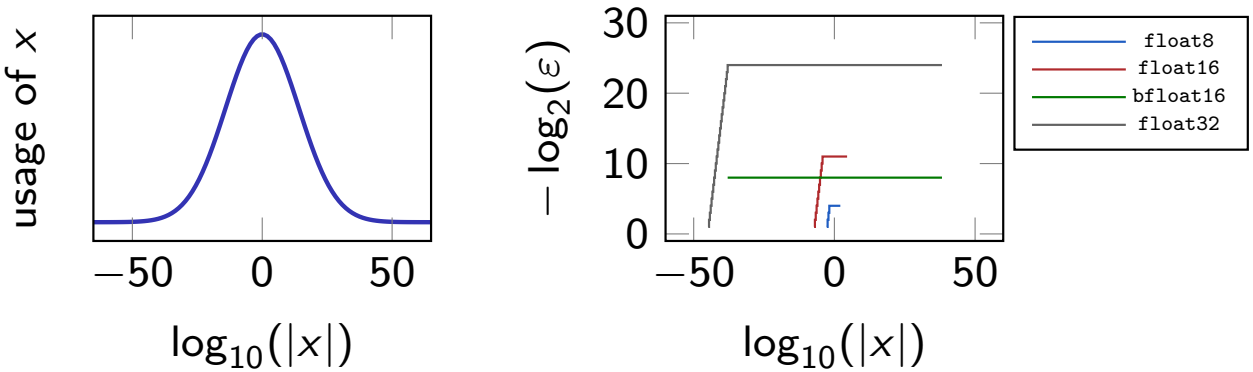

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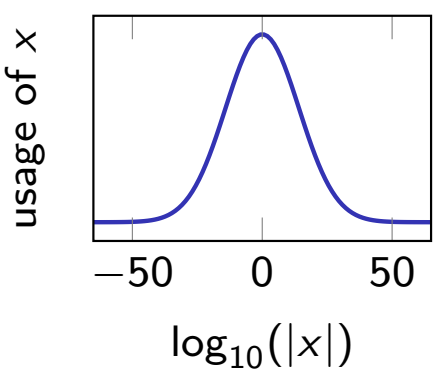

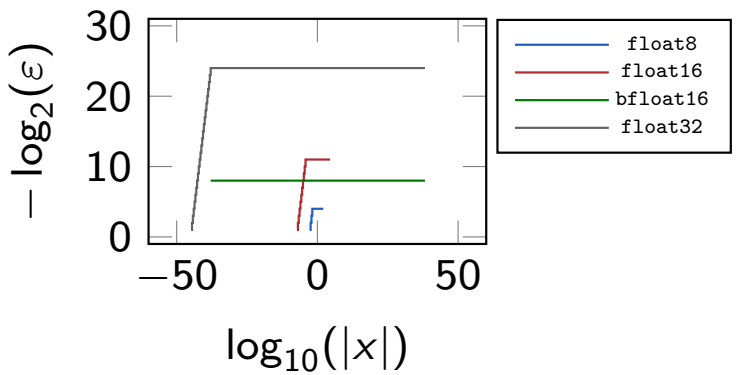

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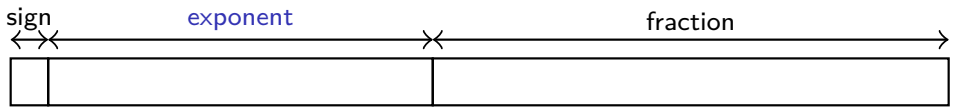

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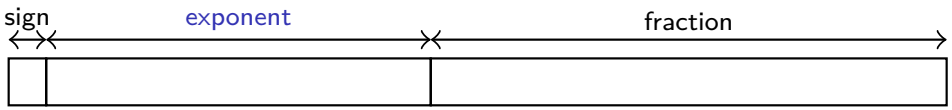

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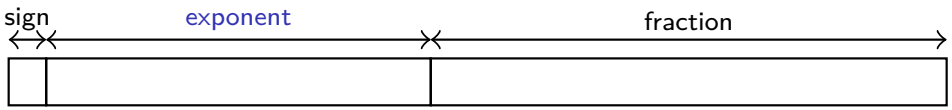

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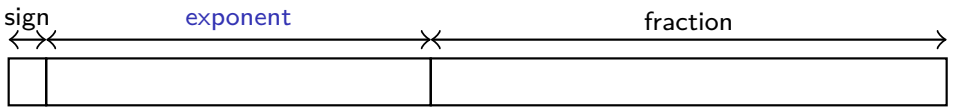

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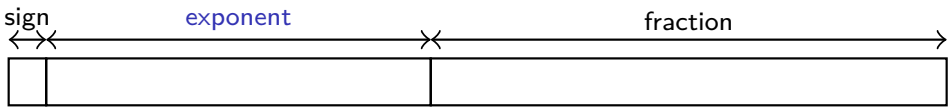

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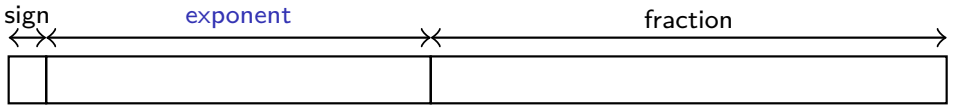

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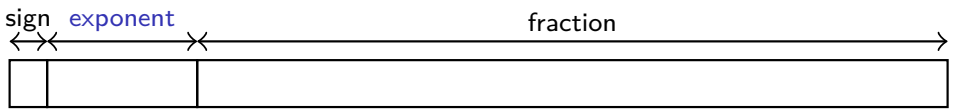

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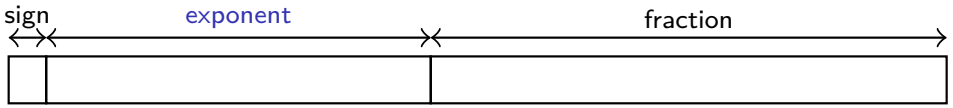

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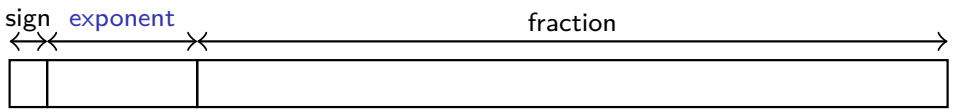

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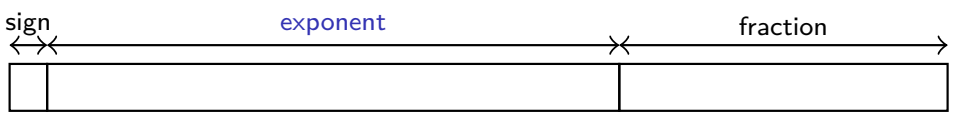

Posit Arithmetic

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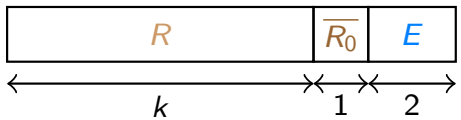

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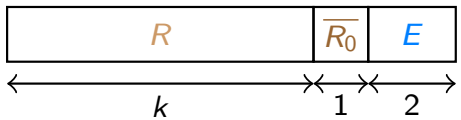

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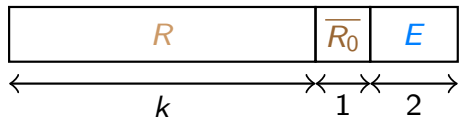

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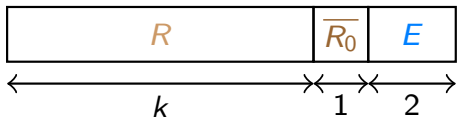

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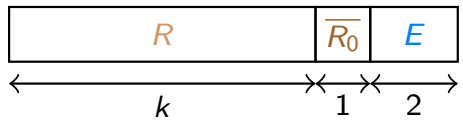

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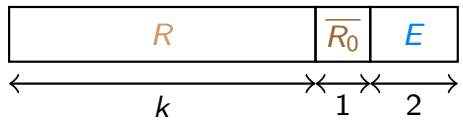

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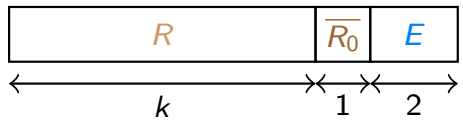

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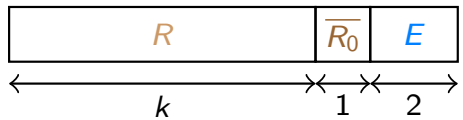

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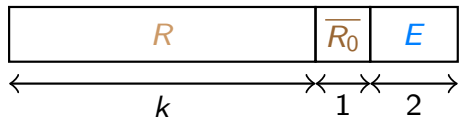

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- Properties
  - No redundant representations
  - Symmetry with two's complement integers

[1] John L. Gustafson et al. 'Standard for Posit Arithmetic (2022)'. Mar. 2022

[2] Florent de Dinechin et al. 'Posits: the good, the bad and the ugly'. Mar. 2019

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- State of the art tapered machine number format, in standardisation[1]
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- Numerous implementations, latest by Calligo with RISC-V SoC 'TUNGA' (2024)
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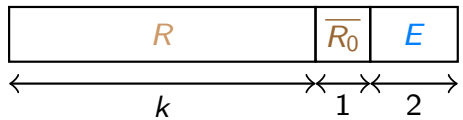

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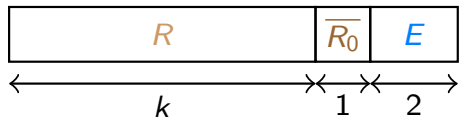

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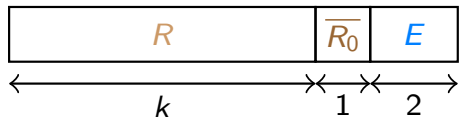

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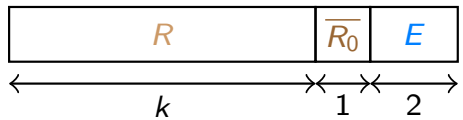

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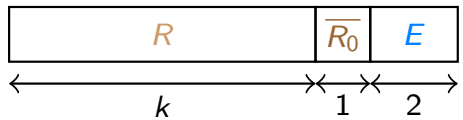

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- Properties
  - No redundant representations
  - Symmetry with two's complement integers (unsigned zero of all zero bits, negation, ordering, sign)
  - Defined for all n, conversion between lengths simple rounding or expansion

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Takum Arithmetic

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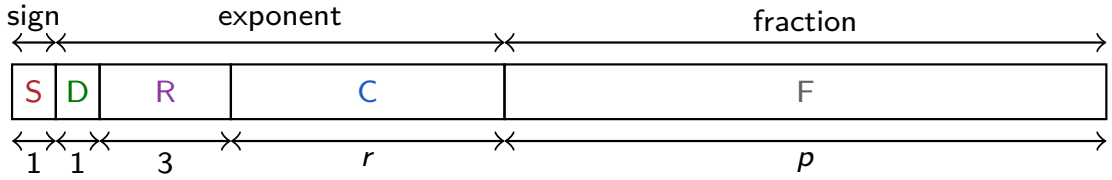
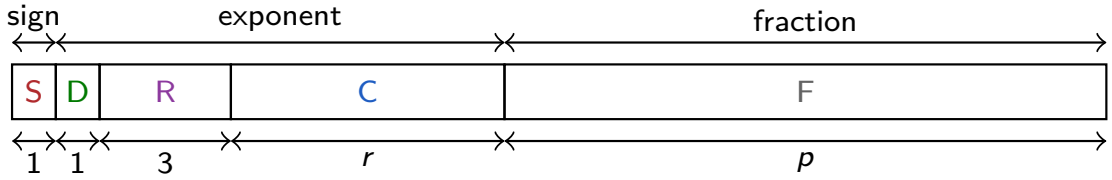


Takum Arithmetic

- Goals



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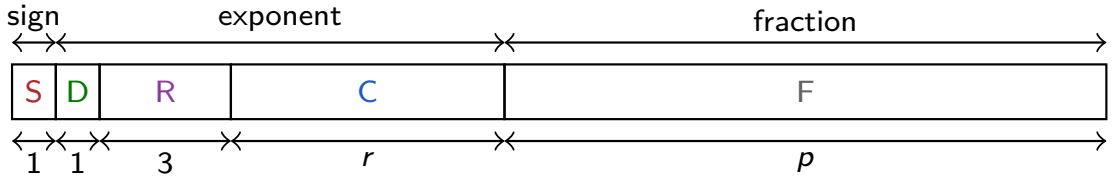

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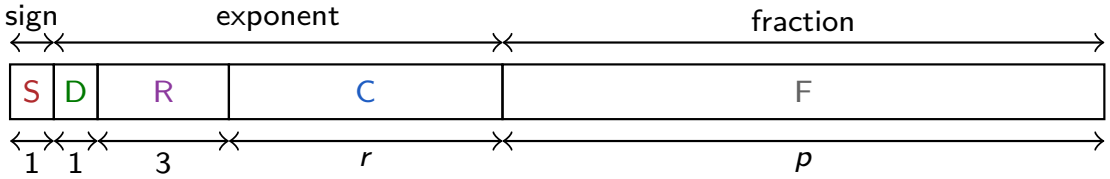
Takum Arithmetic



- Goals
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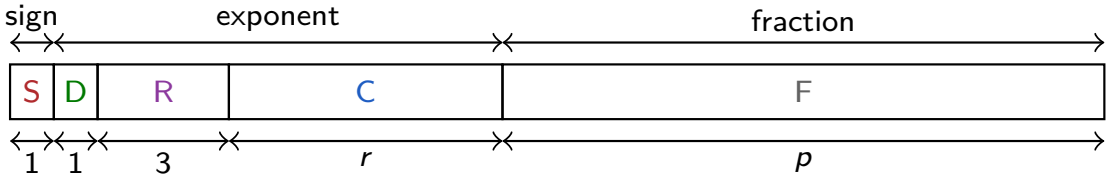
  - Preserve useful posit properties

Takum Arithmetic



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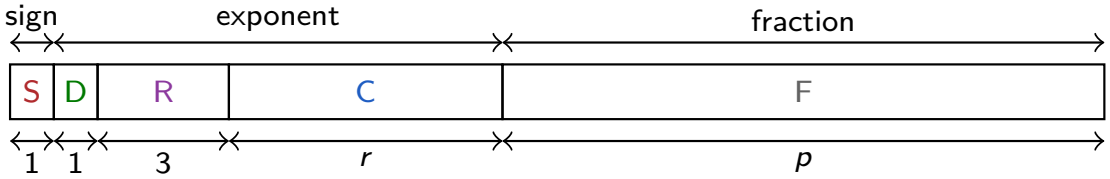
Takum Arithmetic



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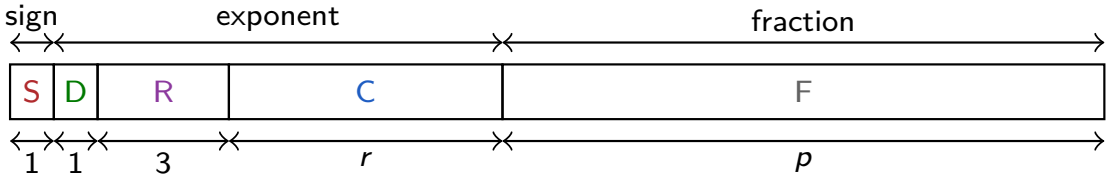

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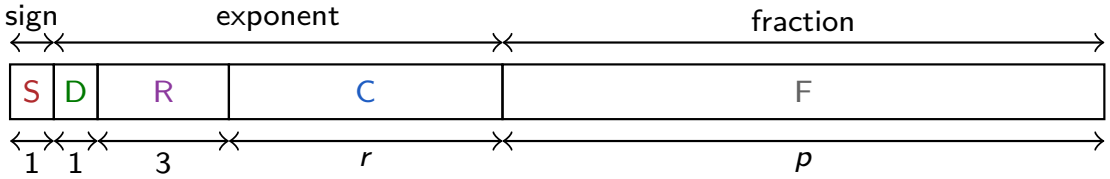

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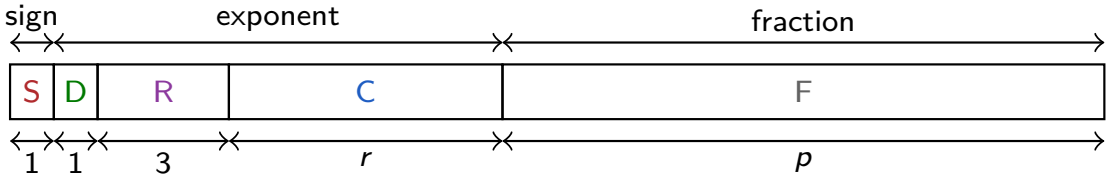

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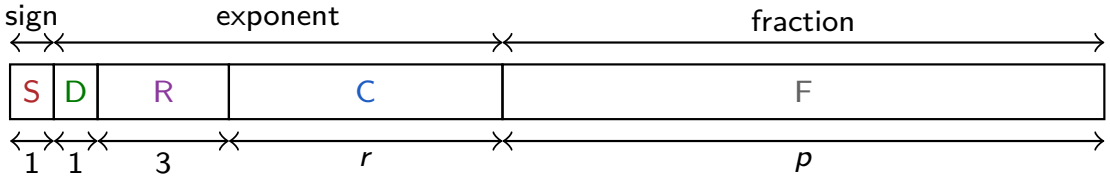

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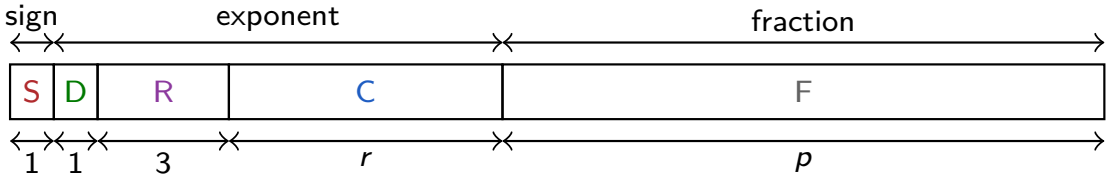

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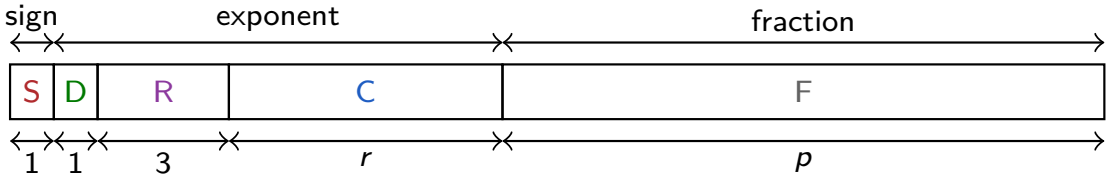

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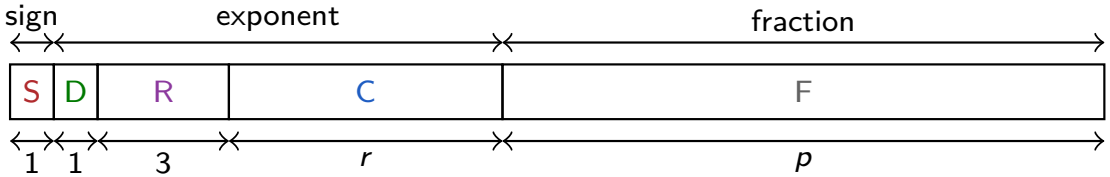
value	0	1	2	3	4	...	254
value bits	1	10	11	100	101	...	11111111
encoding	000	0010	0011	01000	01001	...	1111111111

Takum Arithmetic



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- Icelandic 'takmarkað umfang', meaning 'limited range'

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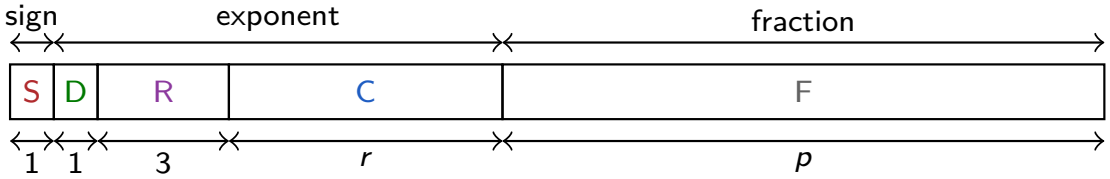

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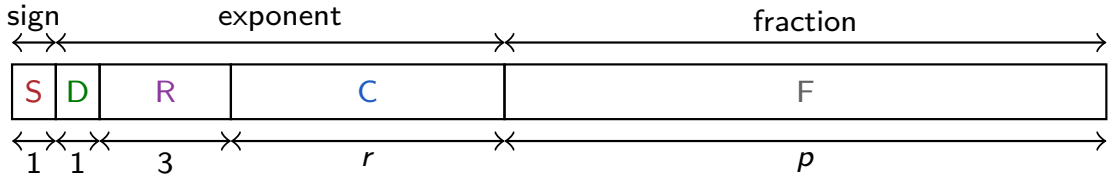
- Icelandic 'takmarkað umfang', meaning 'limited range'
- Separate 'direction' bit D for exponent sign

Takum Arithmetic



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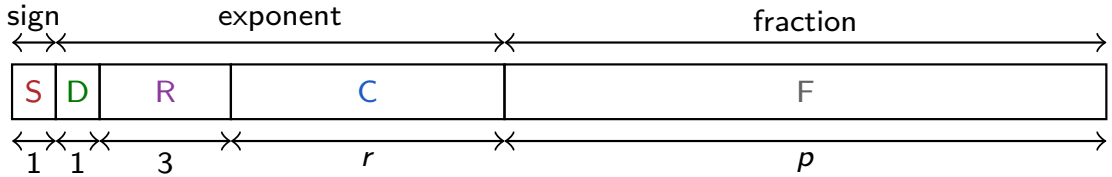
- Interlude: Comparison for value 254 (posit → takum):

Takum Arithmetic



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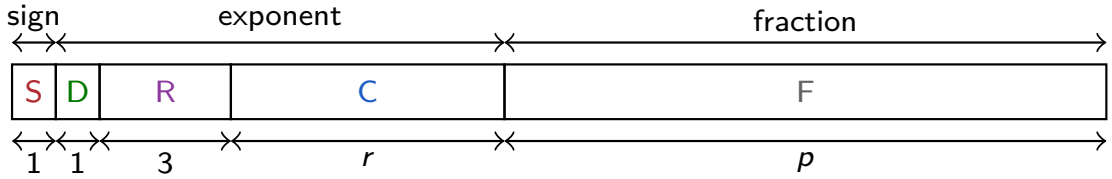
11111111111111111111111111111111111111111111111111111111111111010 (67 bit)

Takum Arithmetic



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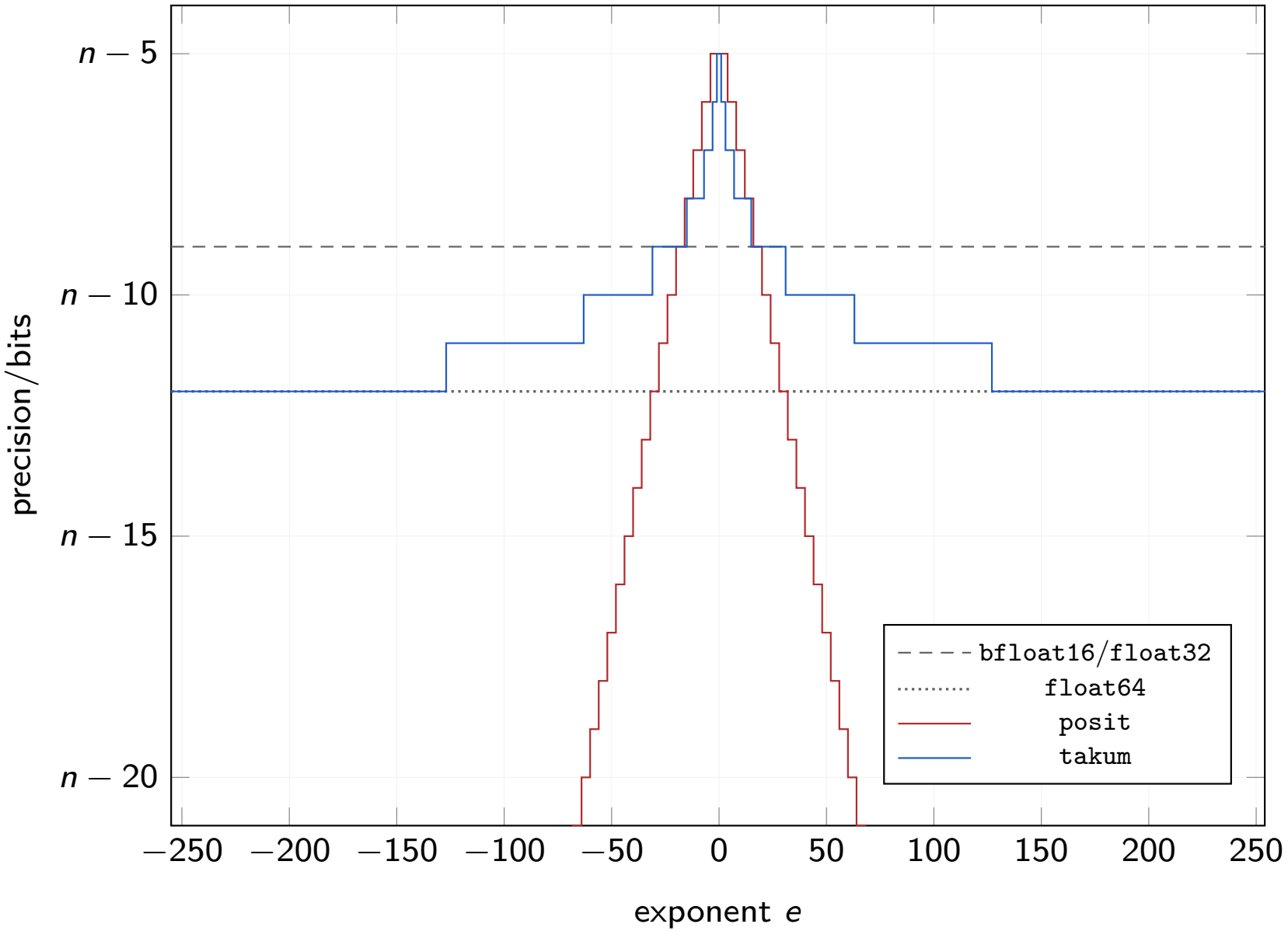
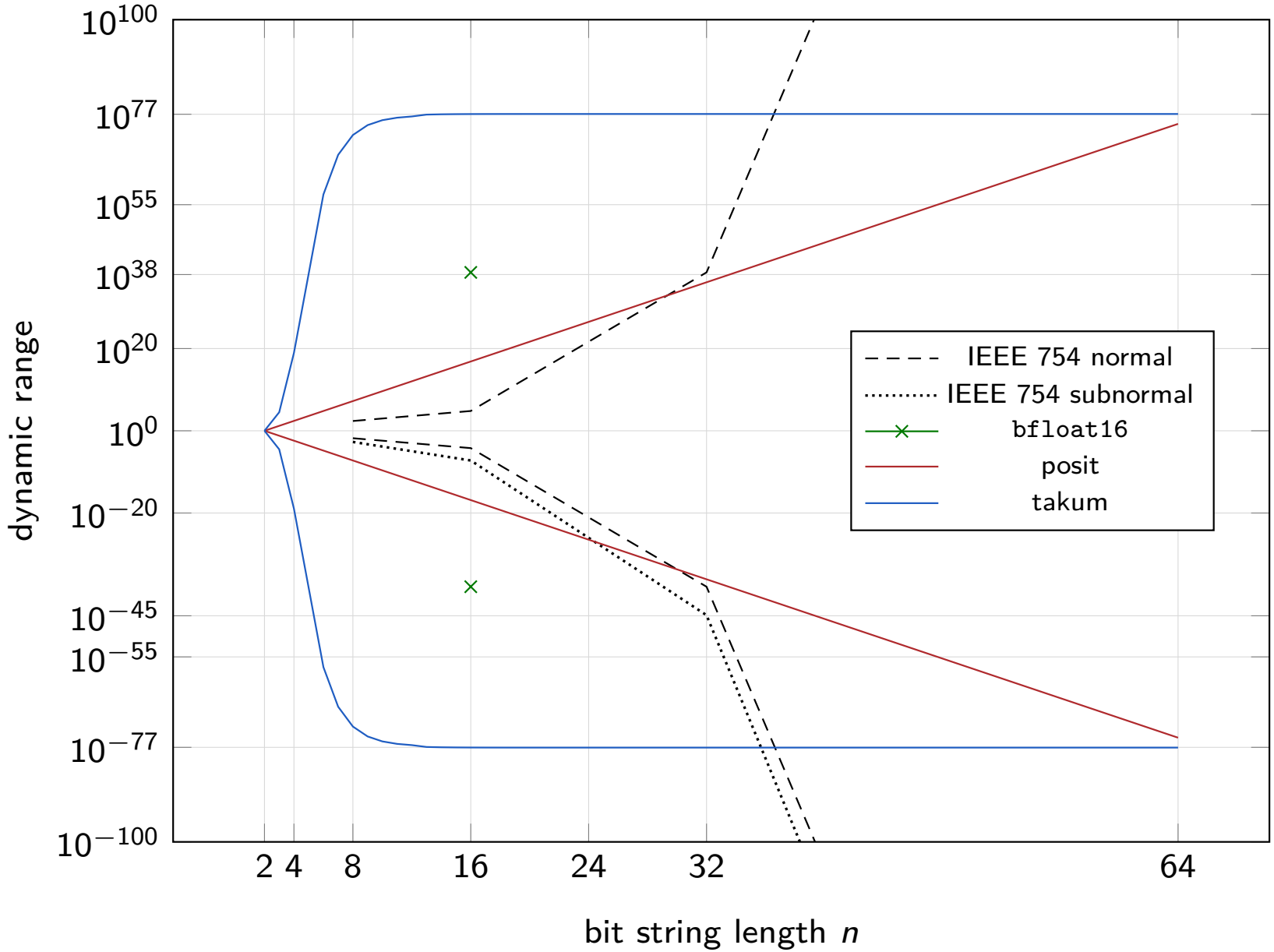
↓

11111111111 (11 bit)

Precision


$n-5$
$n-10$
$n-15$
$n-20$
precision/bits
−250
−200
−150
−100
−50
0
50
100
150
200
250
exponent e
bfloat16/float32
float64
posit
takum

Dynamic Range



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Procedure for each matrix A and type T

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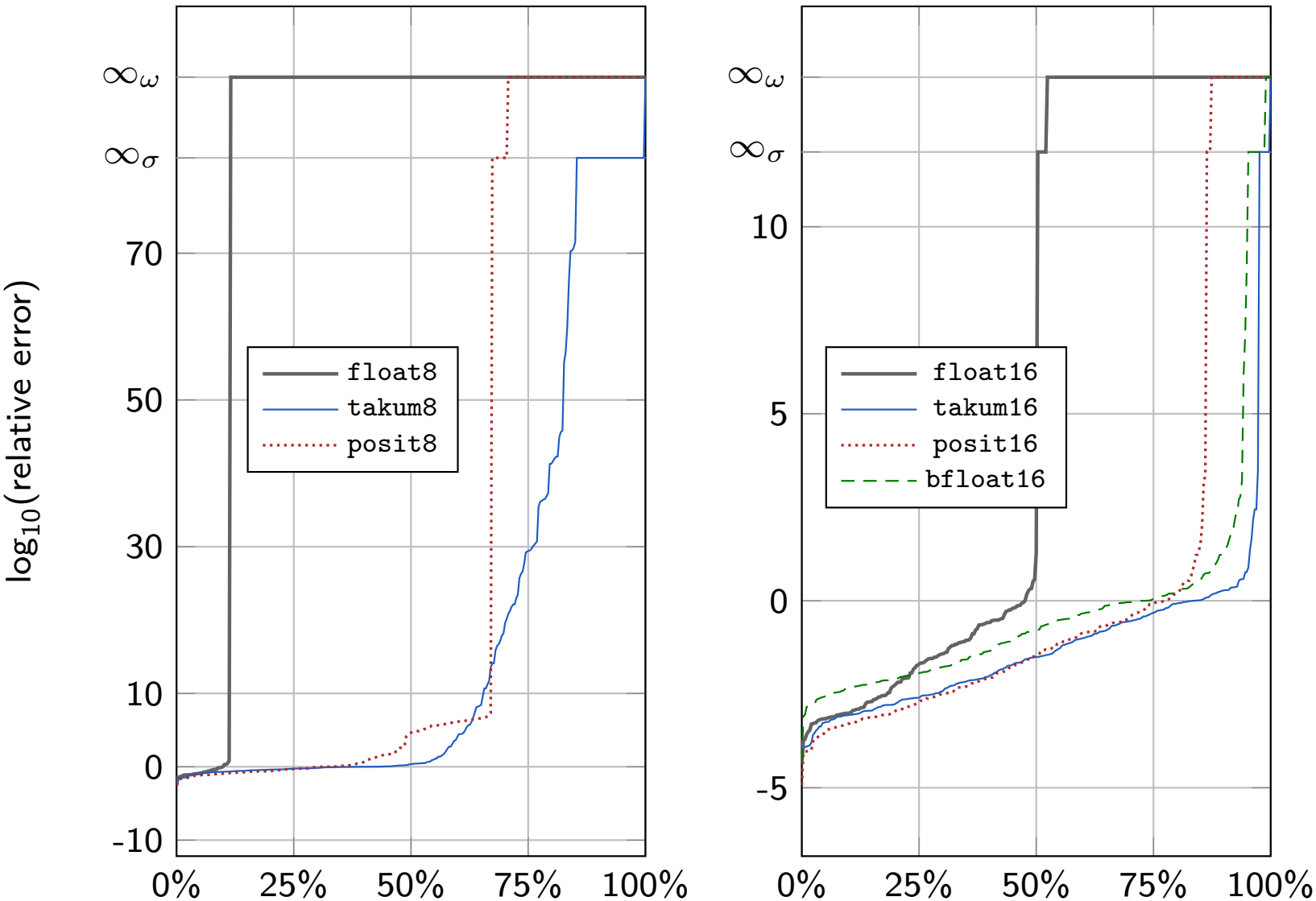
Sort all errors to obtain cumulative error distribution

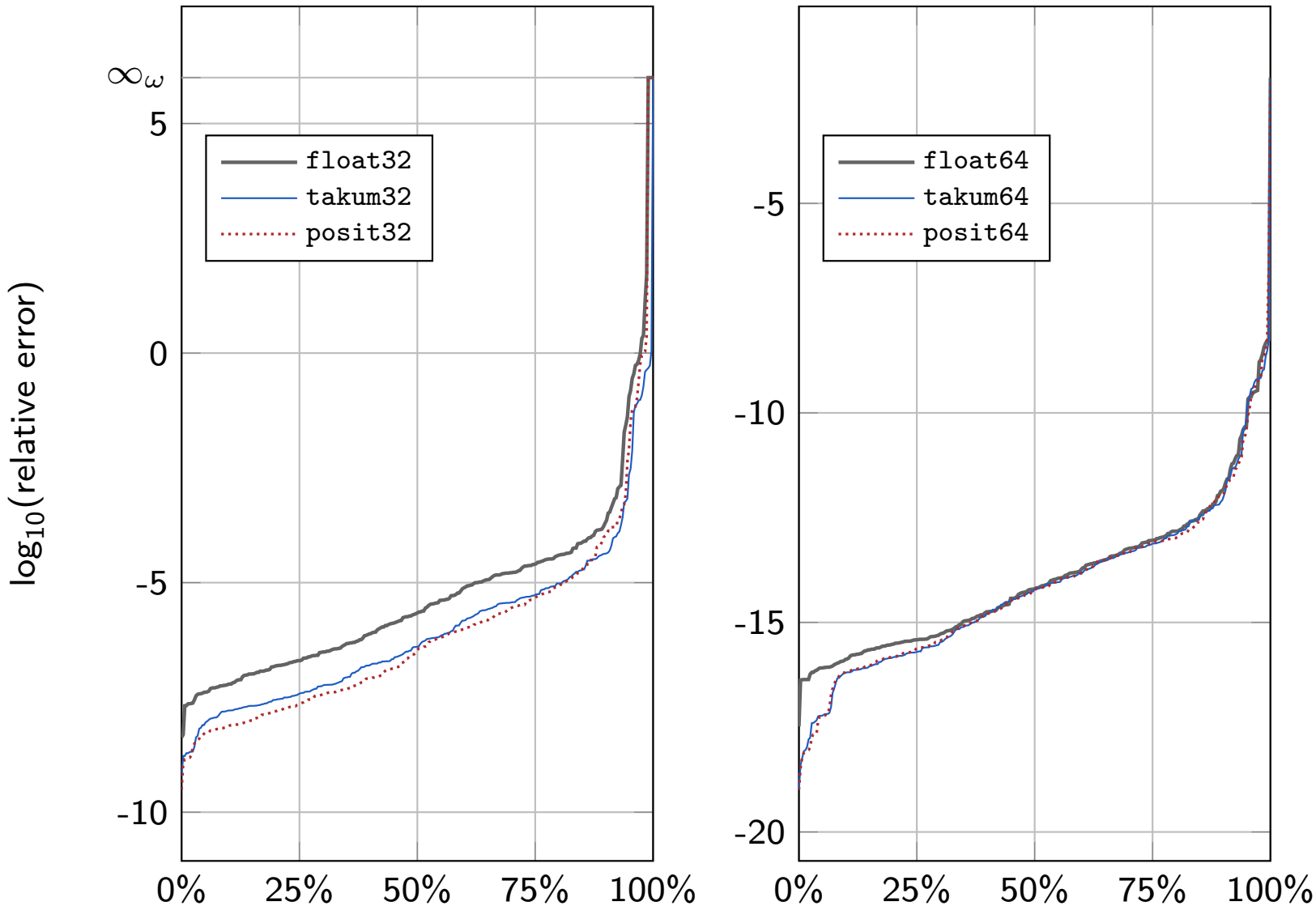
LU (1/2)

8 and 16 bits



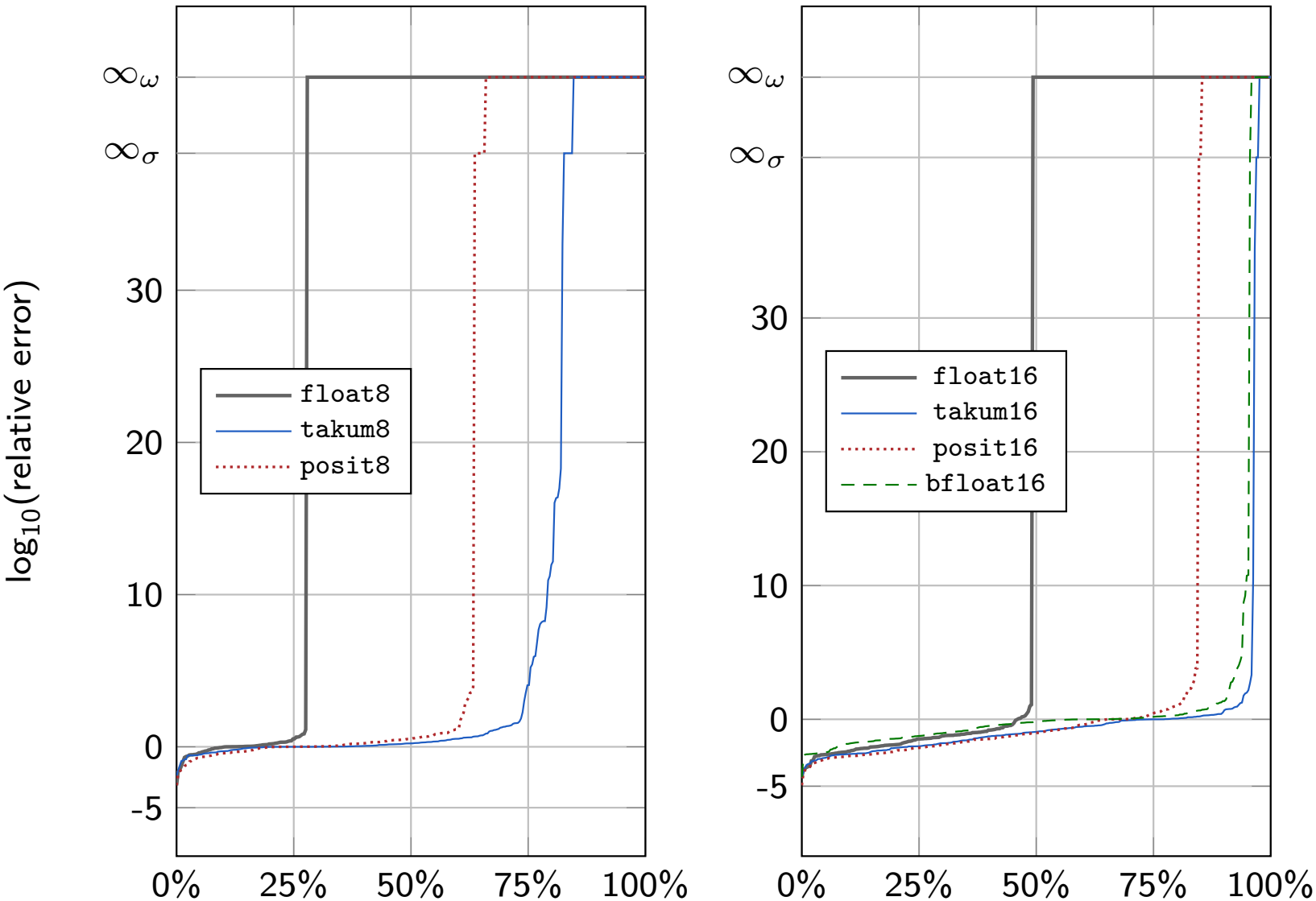
LU (2/2)

32 and 64 bits



QR (1/2)

8 and 16 bits



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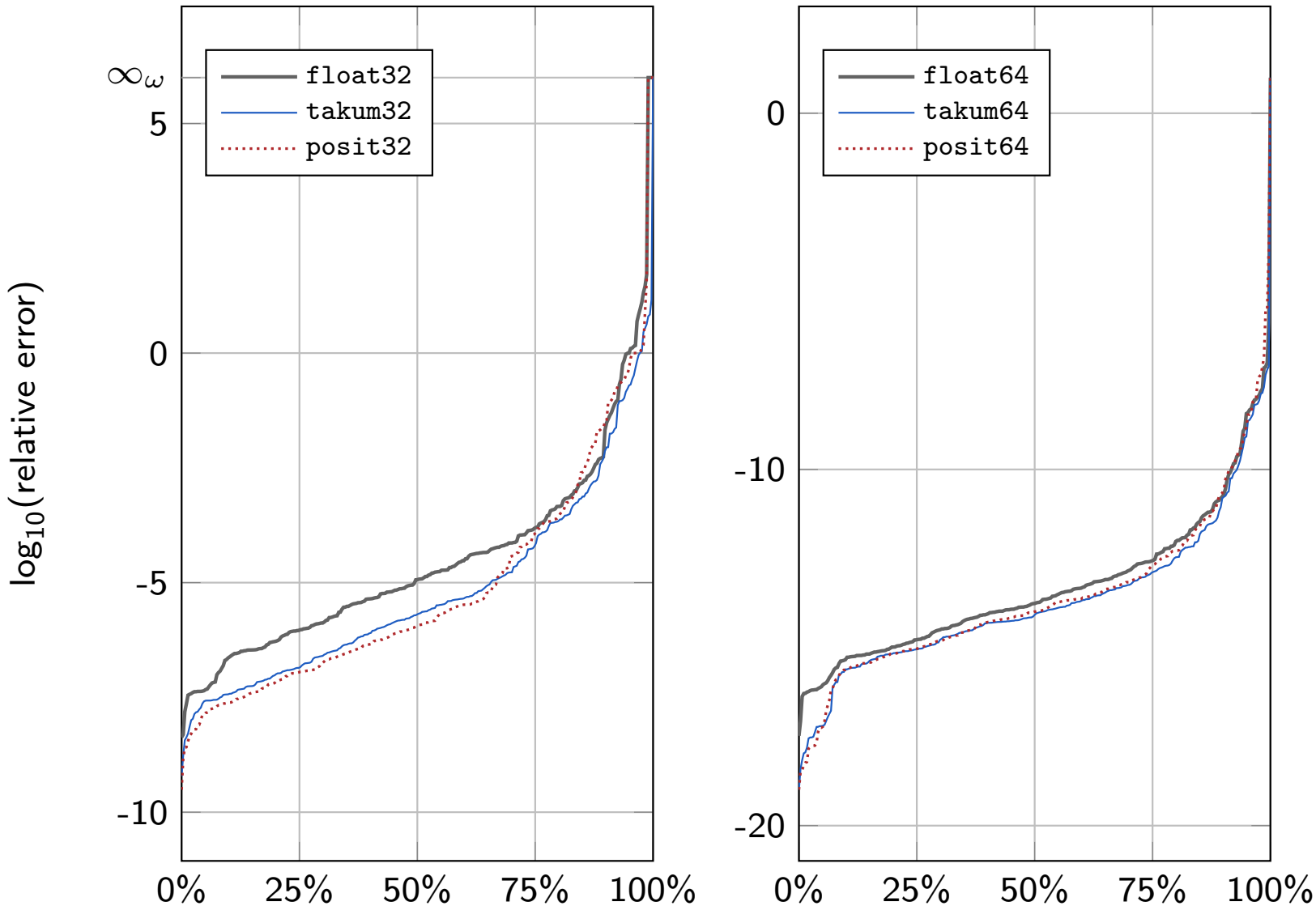
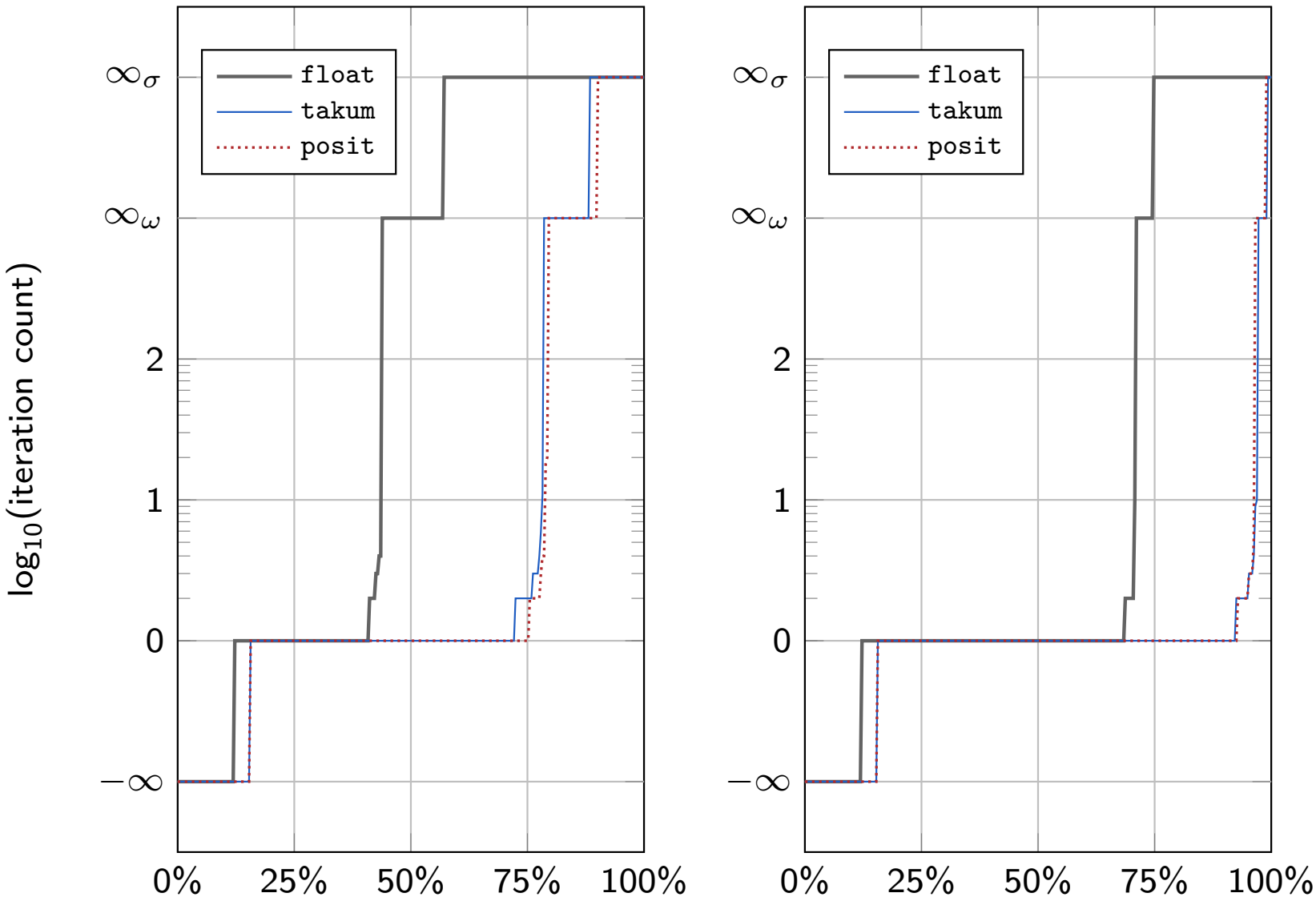


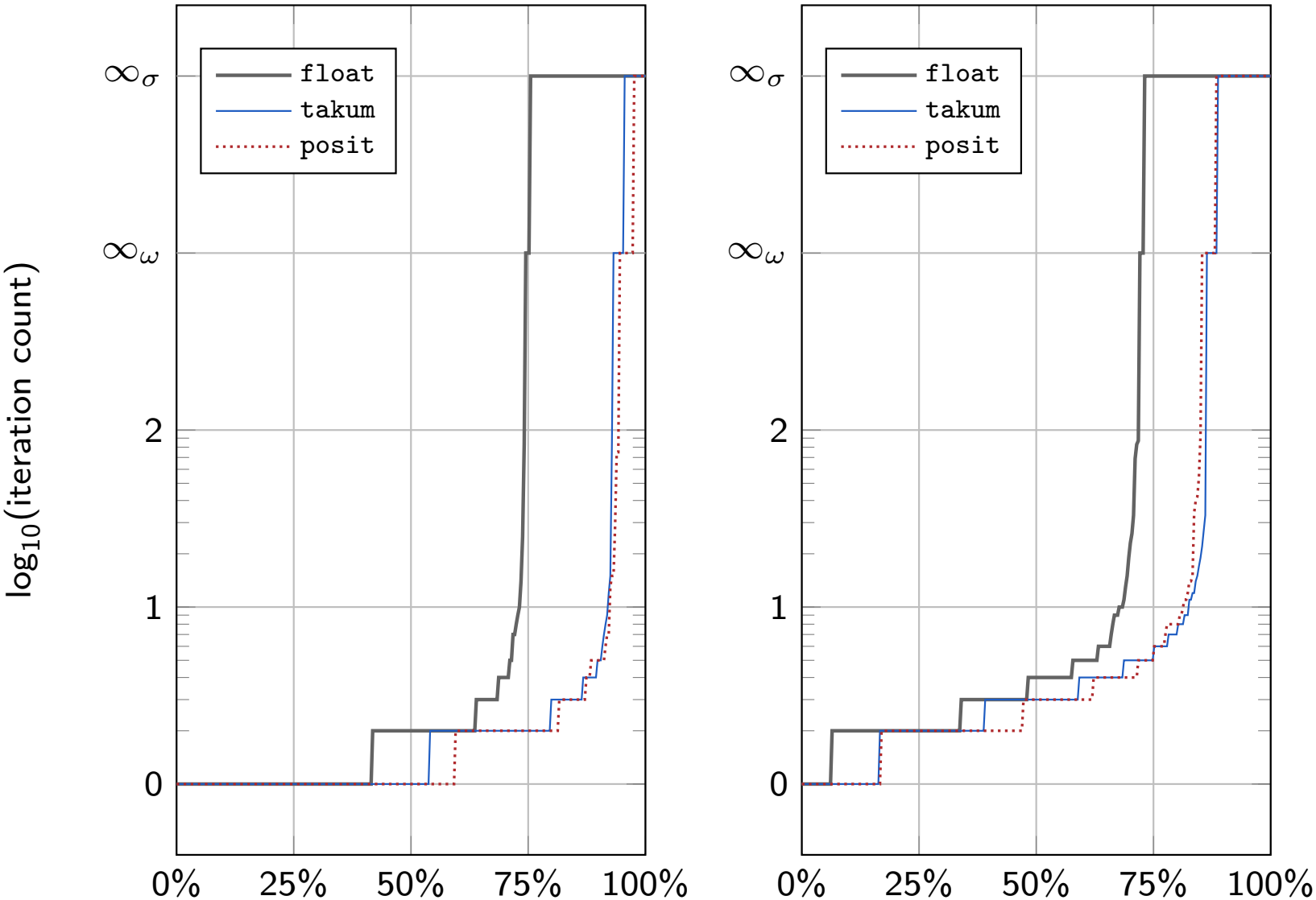
MPIR (1/2)

$(L, W, H) = (8, 16, 32)$ and $(L, W, H) = (16, 16, 32)$, relative tolerance 10^{-3} and 10^{-3}



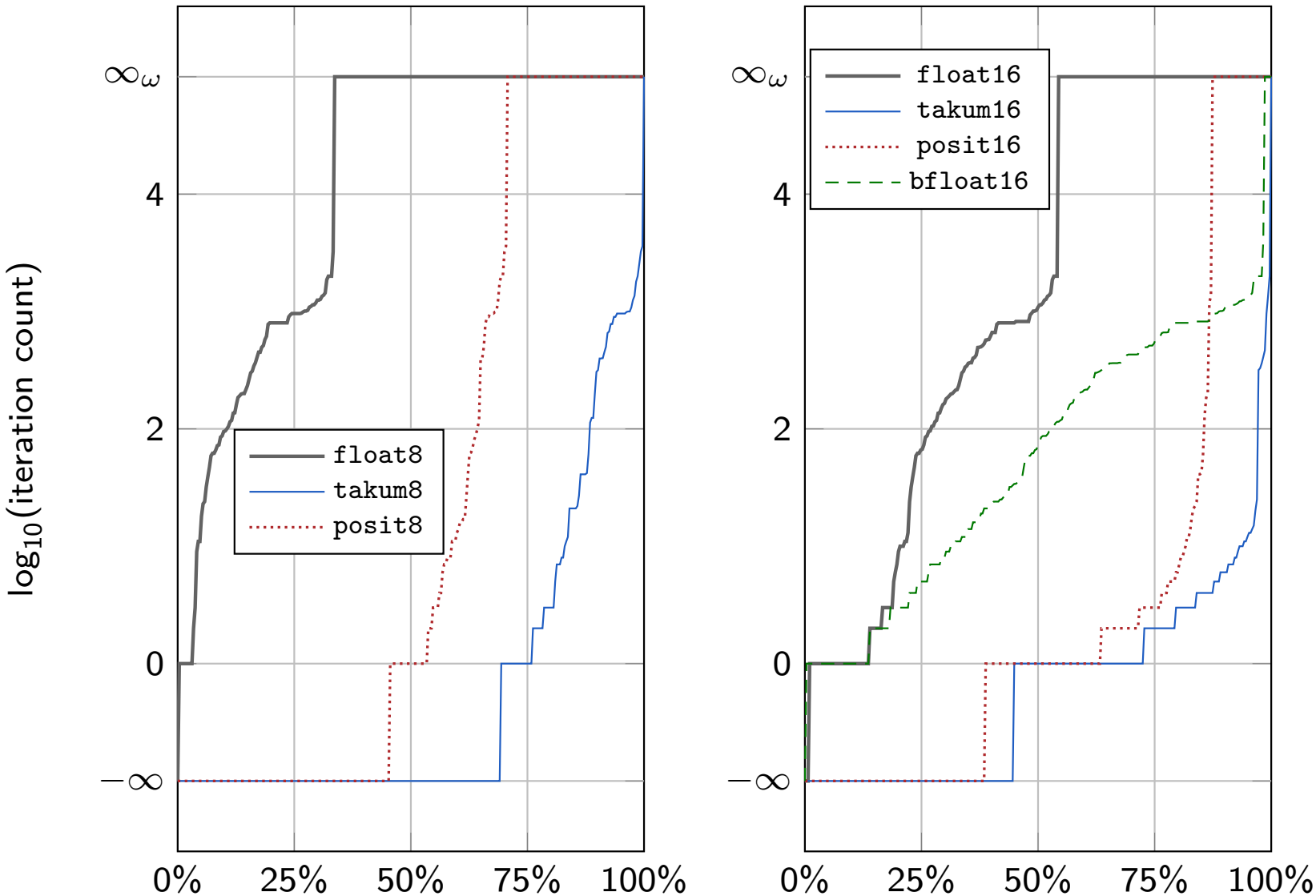
MPIR (2/2)

$(L, W, H) = (16, 32, 32)$ and $(L, W, H) = (16, 32, 64)$, relative tolerance 10^{-6} and 10^{-9}



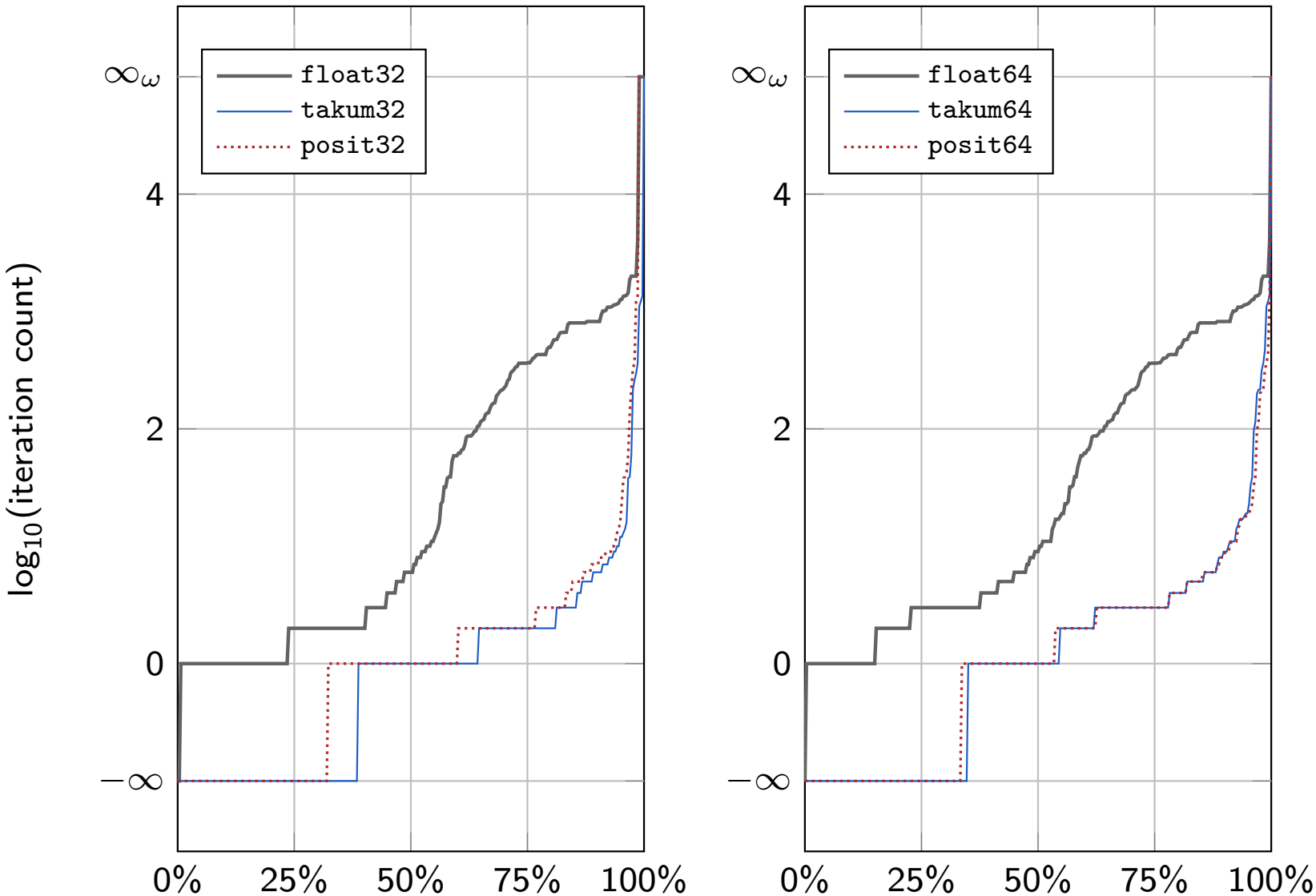
GMRES (1/2)

8 and 16 bits, restart value 20, relative tolerance $\sqrt{\varepsilon_{\texttt{float8}}}$ and $\sqrt{\varepsilon_{\texttt{float16}}}$



GMRES (2/2)

32 and 64 bits, restart value 20, relative tolerance $\sqrt{\varepsilon_{\texttt{float32}}}$ and $\sqrt{\varepsilon_{\texttt{float64}}}$



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- Posits have some shortcomings (precision loss further from 1, limited dynamic range), but overall better than IEEE 754 floats
- Takums suggest new mixed-precision workflow: Reducing n only affects precision, not dynamic range